Two Approaches to Optimal Control of Linearized Navier-Stokes Equations

I. Direct Discretization of the Input to Output Behavior and II. Generalized Riccati Equations

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Séminaire P′

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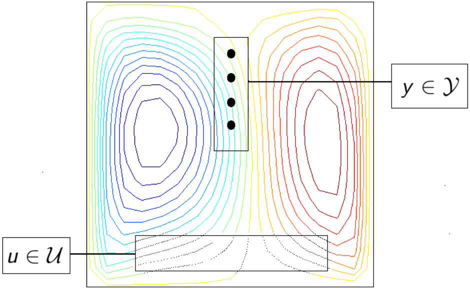
- Physical system
- Influenced by an input $\rightarrow$ $\boxed{u \in \mathcal{U}}$
- Observed via sensors $\rightarrow$ $\boxed{y \in \mathcal{Y}}$ output
- Defines a map: $\boxed{\mathbf{G} : u \mapsto y}$



Motivation for the use of linear I/O maps:

- Input/Output behavior may be comparatively simple
- Control acts local in time

- Bounded linear mappings $\mathbf{G}:\mathcal{U}\to\mathcal{Y}$ can represent linear systems with distributed control.

- For finite dimensional $z(t)$, e.g., for a $z:(0,T]\to\mathbb{R}^N$ satisfying the linear time-invariant system:

$$\begin{aligned} E\dot z(t) - Az(t) &= Bu(t), \text{ on } (0,T], \quad z(0) = z_0 \in \mathbb{R}^N \\ y(t) &= Cz(t) \end{aligned}$$

with an in input $u(t)\in\mathbb{R}^{N_u}$, output $y(t)\in\mathbb{R}^{N_y}$, $E, A\in\mathbb{R}^{N,N}$ and matrices B and C of appropriate size, we have a formula for $\mathbf{G}$:

$$\begin{aligned} E\dot{z}(t) - Az(t) &= Bu(t) \\ y(t) &= Cz(t) \end{aligned}$$

For the finite dimensional case we have for $t \in [0, T]$

$$y(t) = C\Big[\int_0^t e^{E^D A(t-s)} E^D Bu(s)ds + \\ + (I - E^D E) \sum_{i=0}^{\nu-1} (EA^D)^i A^D Bu^{(i)}(t)\Big],$$

provided

- DAE calculus as differentiation index ν and Drazin inverse E^D
- (E, A) is a regular, commuting matrix pair
- u sufficiently smooth and consistent
- $z_0 = 0$

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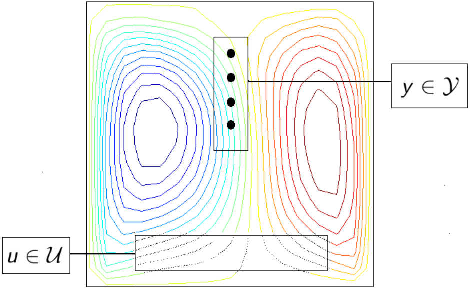


Basic Assumptions and Notation:

- $\mathcal{U}, \mathcal{Y}$ Hilbert-spaces of the signals
- $\mathbf{G} : \mathcal{U} \rightarrow \mathcal{Y}$ a linear input/output map of a given system

Consider finite dimensional subspaces

$$\mathcal{U}_h = \operatorname{span}\{\mu_1, \dots, \mu_p\} \subset \mathcal{U},$$
$$\mathcal{Y}_h = \operatorname{span}\{\nu_1, \dots, \nu_q\} \subset \mathcal{Y},$$

orthogonal projectors $\mathcal{P}_{\mathcal{U}_h}, \mathcal{P}_{\mathcal{Y}_h}$ and signal approximations

$$u_h = \begin{bmatrix} u_1 & \cdots & u_p \end{bmatrix} \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_p \end{bmatrix} = \sum_{j=1}^{p} u_j \mu_j \in \mathcal{U}_h$$

and

$$y_h = \begin{bmatrix} y_1 & \cdots & y_q \end{bmatrix} \begin{bmatrix} \nu_1 \\ \vdots \\ \nu_p \end{bmatrix} = \sum_{i=1}^{q} y_i \nu_i \in \mathcal{Y}_h.$$

We are looking for an approximation $\mathbf{G}_h := \mathcal{P}_{\mathcal{Y}_h}\mathbf{G}\mathcal{P}_{\mathcal{U}_h}$:

- For $u_h \in \mathcal{U}_h$, we have $\mathbf{G}u_h \in \mathcal{Y}$ and

$$\mathcal{P}_{\mathcal{Y}_h}\mathbf{G}u_h = \sum_{i=1}^{q} (\nu_i, \mathbf{G}u_h)_{\mathcal{Y}} \nu_i$$

  provided $\{\nu_i\}$ is an orthogonal basis of $\mathcal{Y}_h$.

- And with $u_h = \sum_{j=1}^{p} u_j \mu_j$ we get

$$\mathbf{G}_h u_h = \sum_{i=1}^{q} \sum_{j=1}^{p} u_j (\nu_i, \mathbf{G}\mu_j)_{\mathcal{Y}} \nu_i$$

.

Thus, we define a finite dimensional approximation to $\mathbf{G}$ via

$$\mathbf{G}_h : \mathcal{U}_h \to \mathcal{Y}_h$$
$$\begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ \vdots \\ y_q \end{bmatrix} = \mathbf{G}_h \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix},$$

with $\mathbf{G}_h = \Big[(\nu_i, \mathbf{G}\mu_j)_{\mathcal{Y}} \Big]_{\substack{i=1,\dots,q \\ j=1,\dots,p}}$.

Lemma

$$\Big[\mathcal{U}_h \to \mathcal{U},\ \mathcal{Y}_h \to \mathcal{Y} \text{ and } \mathbf{G} \text{ bounded} \Big] \quad \Rightarrow \quad \mathbf{G}_h \to \mathbf{G}$$

Linearized Navier-Stokes equation for $\mathfrak{v}$ and $\mathfrak{p}$

$$\begin{aligned}\partial_t \mathfrak{v} + (v_\infty \cdot \nabla)\mathfrak{v} + (\mathfrak{v} \cdot \nabla)v_\infty + \nabla \mathfrak{p} - \frac{1}{Re}\triangle \mathfrak{v} =& (v_\infty \cdot \nabla)v_\infty + \mathfrak{B}u,\\ \nabla \cdot \mathfrak{v} =& 0,\\ \mathfrak{y} =& \mathfrak{C}\begin{bmatrix}\mathfrak{v}\\ \mathfrak{p}\end{bmatrix}.\end{aligned}$$

Parameters: Re and reference velocity $\mathfrak{v}_\infty$

Spatial discretization yields a descriptor system

$$\begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}\frac{d}{dt}\begin{bmatrix}v\\ p\end{bmatrix} - \begin{bmatrix}A & J_1^T\\ J_2 & Q\end{bmatrix}\begin{bmatrix}v\\ p\end{bmatrix} = \begin{bmatrix}Bu\\ 0\end{bmatrix},$$

$$y = C\begin{bmatrix}v\\ p\end{bmatrix}.$$

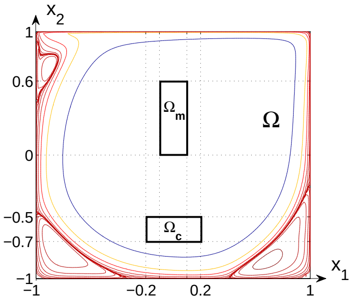
$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v \\ p \end{bmatrix} - \begin{bmatrix} A & J_1^T \\ J_2 & Q \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} = \begin{bmatrix} Bu \\ 0 \end{bmatrix}$$

If $\begin{bmatrix} A - \lambda I & J_1^T \\ J_2 & Q \end{bmatrix}$ is invertible for a $\lambda \in \mathbb{C}$, we can apply the solution formula for linear descriptor systems.

In particular, the corresponding I/O map $\mathbf{G}$ is well-defined if

Necessary Regularity of the Input Signals

- u differentiable (general case) or
- u continuous, if $J_1 = J_2$ and Q is a minimal stabilization, or the output does not depend on the pressure p



Test Case - Driven cavity

- Modelled by Navier-Stokes Equations linearized about the steady state solution v_0
- $Re = 3333$, $Q1$-$P0$ mixed finite elements on uniform 256×256 grid [IFISS]
- Simulation interval $(0, 0.1]$ starting from v_0
- 2D input signal in domain of control Ω_c
- $y(t)$ - velocity in domain of observation Ω_m, spatially avaraged in x_1-direction

Problem: given a target output $y^* \in \mathcal{Y}$, find $u^* \in \mathcal{U}$ that solves:

$$\|y^* - \mathbf{G}u\|_{\mathcal{Y}}^2 + \alpha\|u\|_{\mathcal{U}}^2 \to \min$$

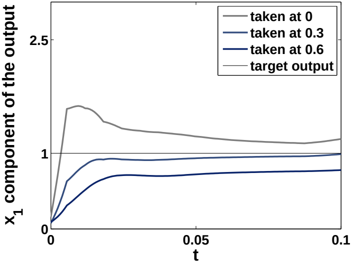
Approach: Use the matrix approximation $\mathbf{G}_h$ to find an approximation $u_h^* \in \mathcal{U}_h$

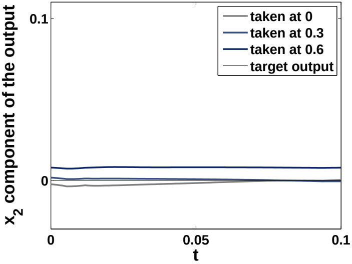
Testcase Driven Cavity

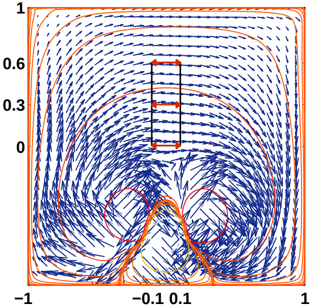
- $\dim \mathcal{U}_h = \dim \mathcal{Y}_h = 34 \cdot 16$
  - 34 piecewise linear (space) $\times$ 16 Haar wavelets (time)

$\longrightarrow$ $34 \cdot 16$ offline solves: $\sim$ 10h on desktop PC

$\longrightarrow$ Online time to solve the optimization problem: 0.034s

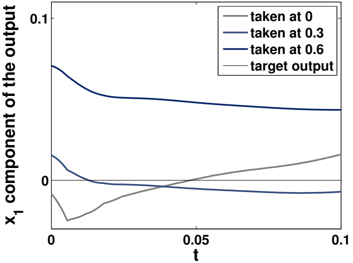


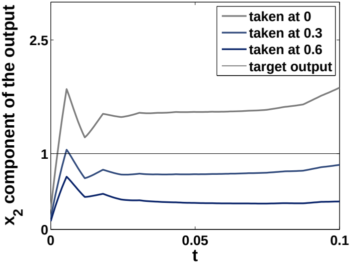


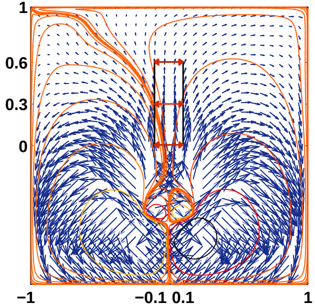


Application example for

$$y^* \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$







Application example for

$$y^* \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Some final remarks

- Approach seems feasible for distributed control of linearized flow equations
  $\longrightarrow$ Applicability in iterative schemes

- Further model reduction by higher order SVD succesfully tested for heat conduction

- And a referee wanted to know ...
  "...what is the gain if compared to the standard approach"?

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This is actually three questions:

1. What is the standard approach?
   For linear systems? LQR - **L**inear-**Q**uadratic **R**egulator!
2. What is the gain?
   It's computable and very fast in the online computation.
3. How do both approaches compete?
   No answer here, since the LQR (and other standard methods for LQ problems) are not feasible for the chosen setup.

The Linear Quadratic Programming Problem:

Find an input u, such that a quadratic cost specification

becomes minimal – subject to a linear system.

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$$\mathcal{J}(y,u) := y(T)^T \mathcal{M} y(T)^T + \int_0^T y(t)^T W y(t) + u(t)^T R u(t) dt$$

becomes minimal – subject to a linear system

$$\begin{aligned} E\dot{z}(t) - Az(t) &= Bu(t), \text{ on } (0,T), \quad z(0) = z_0, \\ y(t) &= Cz(t), \end{aligned}$$

with weighting matrices $\mathcal{M}$, W and R symmetric positive definite and R invertible.

$$\mathcal{J}(y,u) \to \min_u$$
s.t. a linear system

1. What gives the gain?
   - The system is replaced $\mathcal{G} : u \mapsto y$
   - and the optimization becomes unconstrained

$$\mathcal{J}(y,u), \text{s.t. a system} \to \mathcal{J}(Gu,u) =: \hat{\mathcal{J}}(u)$$

   - Now optimization is comparatively easy, in particular in finite dimensions

But in general G is not readily available

$$\mathcal{J}(y,u) \to \min_u$$
$$\text{s.t. } E\dot{z}(t) - Az(t) - Bu(t) = 0$$

2. Standard approaches solve the optimality system on (0,T):

$$\begin{aligned} E\dot{z} - Az - Bu &= 0, \quad z(0) = z_0, \\ -E^T\dot{\lambda} - A^T\lambda + Wz &= 0, \quad \lambda(T) = -\mathcal{M}z(T), \qquad \text{(adjEqn)} \\ -B^T\lambda + Ru &= 0, \end{aligned}$$

$\mathcal{M}, W, R$ from the cost functional,
e.g. via the Riccati approach

Standard case (E=I):
The above optimality system has a unique solution and the optimal input u is given via the feedback law $u = R^{-1}B^TPz$, where P is the unique solution to the Riccati equation:
$\dot{P} + A^TP + PA - PBR^{-1}B^TP - W = 0.$

... s.t. $E\dot{z}(t) - Az(t) - Bu(t) = 0$

Our case, for the sake of brevity $y := v$ and $J_1 = J_2 = J$:

Minimize

$$\mathcal{J}(v,u) := v(T)^T \mathcal{M} v(T)^T + \int_0^T v(t)^T W v(t) + u(t)^T R u(t) dt$$

subject to linearized Navier-Stokes equations

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{p} \end{bmatrix} - \begin{bmatrix} A & J^T \\ J & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} - \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad v(0) = v_0.$$

Note that, "E" $= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible.

3. Why the standard approaches do not work?
   - DAE-structure: "E" is singular
     - one may use Helmholtz projection to reformulate the NSE as a standard ODE
     - this is not very feasible for large systems and numerically unstable
   - High-dimensionality
     - For the 2D driven cavity example, the optimality system is a boundary value problem of size 10^6
     - e.g., a solution by finite differences on N discrete time instances, leads to a system of $N * 10^6$ equations

Our proposal to tackle this problem:

- keep the DAE structure, as it ensures stability and physical validity of the solution
- Generalize the LQR approach to the system class under consideration
- Hope for efficient algorithms for the solution of large-scale Riccati equations, that are investigated in the group of Peter Benner at the Max-Planck Institute in Magdeburg, Germany

Voilá, with the Riccati decoupling

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} X & Y^T \\ Y & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix}$$

we can state that

Lemma:

The optimal input is given by $u = R^{-1}B^TXv$, where X is the unique solution of

$$\dot{X} + A^TX + XA + XB^TR^{-1}BX - \\ -W + J^TY + Y^TJ = 0,$$

$$XJ^T = 0, \quad JX = 0,$$

$$X(T) = -\mathcal{M}.$$

What have we got – we haven't solved the optimization problem:

- The problem is no more a boundary value problem, but an initial value problem
- The problem is made accessible for so called low-rank ADI algorithms, investigated in the group of Benner
- Having kept the DAE structure, we can numerically control the validity of the solution
- If one solves the problem, one obtains not only the solution u but the optimal feedback law

Summary:

- Optimal control of linearized Navier-Stokes equation
- Direct discretization of the corresponding I/O behavior
- Standard LQ-approaches are not applicable
- A possible generalization is obtained via a Riccati DAE ansatz
- Future task: numerical solution of the Riccati DAE
- Future task: application in nonlinear problems

Thanks to Volker Mehrmann and

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For suggestions and questions please contact me

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